9.3.09

Analysis UE

I, 372, 381, 387, 390, 392, 396, 401

372) $f(x) = x^{\frac{2}{3}} - (x^2-1)^{\frac{1}{3}}$

a^r ist undef. $\forall a<0,\ r\in\mathbb{Q} \Rightarrow D(f) = [1,\infty)$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}(x^2-1)^{-\frac{2}{3}}\cdot 2x$$
$$= \frac{2}{3}\left(x^{-\frac{1}{3}} - x(x^2-1)^{-\frac{2}{3}}\right)$$
$$f''(x) = \frac{2}{3}\left(-\frac{1}{3}x^{-\frac{4}{3}} - (x^2-1)^{-\frac{2}{3}} - x\left(-\frac{2}{3}\right)(x^2-1)^{-\frac{5}{3}}\cdot 2x\right)$$
$$= \frac{2}{3}\left(-\frac{1}{3}x^{-\frac{4}{3}} - (x^2-1)^{-\frac{2}{3}} + \frac{4x^2}{3}(x^2-1)^{-\frac{5}{3}}\right)$$

f konvex $\Leftrightarrow f''(x) \geq 0$ $\quad |\cdot\frac{3}{2}$

$\Leftrightarrow -\frac{1}{3}x^{-\frac{4}{3}} - (x^2-1)^{-\frac{2}{3}} + \frac{4x^2}{3}(x^2-1)^{-\frac{5}{3}} \geq 0$ $\quad |\cdot 3$

$\Leftrightarrow 4x^2(x^2-1)^{-\frac{5}{3}} - 3(x^2-1)^{-\frac{2}{3}} \geq x^{-\frac{4}{3}}$ $\quad |\cdot(x^2-1)^{\frac{5}{3}}$ (immer ≥ 0 für $x\geq 1$)

$\Leftrightarrow 4x^2 - 3(x^2-1) \geq (x^2-1)^{\frac{5}{3}}\, x^{-\frac{4}{3}}$

$\Leftrightarrow 4x^2 - 3x^2 + 3 = x^2+3 \geq \frac{(x^2-1)^{\frac{5}{3}}}{x^{+\frac{4}{3}}}$

$\Leftrightarrow \frac{(x^2-1)^{\frac{5}{3}}}{x^{\frac{4}{3}}} \leq \frac{(x^2)^{\frac{5}{3}}}{x^{\frac{4}{3}}} = x^2 \leq x^2+3 \qquad \forall x\in D(f)$

also f konvex auf ganz $D(f) \Rightarrow \nexists$ Wendepunkt

381) $\lim_{x\to\infty}\left(\frac{\pi}{2} - \arctan x\right)^{\frac{1}{\ln x}} = \lim_{x\to\infty} e^{\frac{1}{\ln x}\ln(\operatorname{arccot} x)} = e^{\lim_{x\to\infty}\frac{\ln(\operatorname{arccot} x)}{\ln x}}$

$$\lim_{x\to\infty}\frac{\ln(\operatorname{arccot} x)}{\ln x} \overset{\text{Hospital } \frac{?}{\infty}}{=} \lim_{x\to\infty}\frac{\frac{1}{\operatorname{arccot} x}\cdot\frac{-1}{1+x^2}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{-x}{1+x^2}}{\operatorname{arccot} x}$$

$$\overset{\text{Hospital } \frac{0}{0}}{=} \lim_{x\to\infty}\frac{\frac{-(1+x^2)-(-x)2x}{(1+x^2)^{\cancel{2}}}(\cdots 1)}{\cancel{\frac{-1}{1+x^2}}} = \lim_{x\to\infty}\frac{1-x^2}{1+x^2}$$

$$\overset{\text{Hospital } \frac{?}{\infty}}{=} \lim_{x\to\infty}\frac{-2x}{2x} = -1$$

$\Rightarrow \lim_{x\to\infty}\left(\frac{\pi}{2} - \arctan x\right)^{\frac{1}{\ln x}} = e^{-1}$

387) $\sum_{n=1}^{\infty} \ln\left|\frac{x+n}{x+n+1}\right| = \lim_{k\to\infty} \sum_{n=1}^{k} \ln\left|\frac{x+n}{x+n+1}\right|$

$$= \lim_{k\to\infty} \ln \prod_{n=1}^{k} \left|\frac{x+n}{x+n+1}\right|$$

$$= \lim_{k\to\infty} \ln\left(\left|\frac{x+1}{\cancel{x+2}}\right| \cdot \left|\frac{\cancel{x+2}}{\cancel{x+3}}\right| \cdot \ldots \cdot \left|\frac{\cancel{x+k}}{x+k+1}\right|\right)$$

$$= \lim_{k\to\infty} \ln\left|\frac{x+1}{x+k+1}\right| = \ln\Big(\underbrace{\lim_{k\to\infty} \frac{x+1}{x+k+1}}_{\to 0}\Big) = -\infty$$

$\Rightarrow$ Reihe divergiert $\forall x \in \mathbb{R}$ gegen $-\infty$!

390) $f(x) = \sqrt{2+x^{10}} = \sqrt{2} \cdot \sqrt{1+\frac{x^{10}}{2}}$

$$= \sqrt{2} \cdot \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{x^{10}}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2^n} \binom{\frac{1}{2}}{n} x^{10n} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$\Rightarrow f^{(n)}(0) = 0 \quad \forall n \neq 10k,\ k \in \mathbb{N} \cup \{0\}$

also $f^{(33)}(0) = 0$

sonst: $\frac{f^{(10k)}(0)}{(10k)!} = \frac{\sqrt{2}}{2^k} \binom{\frac{1}{2}}{k}$

$$\Rightarrow f^{(20)}(0) = 20!\, \frac{\sqrt{2}}{2^2} \binom{\frac{1}{2}}{2}$$

$$= 20!\, \frac{\sqrt{2}}{4} \cdot \frac{1}{2!} \prod_{i=1}^{2} \left(\frac{3}{2} - i\right)$$

$$= \frac{20!}{4\sqrt{2}} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{20!}{16\sqrt{2}}$$

396) $f(x) = \dfrac{\sqrt{|x+1|}}{x^2+4}$

$$|x+1| = \begin{cases} x+1 & x \geq -1 \\ -(x+1) & x \leq -1 \end{cases}$$

$$\Rightarrow |x+1|' = \begin{cases} +1 & x > -1 \\ -1 & x < -1 \\ \nexists & x = -1 \quad (!) \end{cases}$$

also $f(x)$ diffbar auf $\mathbb{R}\setminus\{-1\}$

<u>Betrachte $x \in \mathbb{R}\setminus\{-1\}$:</u>

$$f'(x) = \frac{\frac{1}{2}|x+1|^{-\frac{1}{2}}(\pm 1)(x^2+4) - |x+1|^{\frac{1}{2}}2x}{(x^2+4)^2}$$

$$= \frac{\pm(x^2+4) - 4x|x+1|}{2\sqrt{|x+1|}(x^2+4)^2} = \pm \frac{-3x^2-4x+4}{2\sqrt{|x+1|}(x^2+4)^2}$$

$$f'(x) = 0 \iff 3x^2+4x-4 = 0$$

$$\iff x_{1,2} = \frac{-4 \pm \sqrt{16+48}}{6} = \frac{-4 \pm 8}{6}$$

$$\iff x_1 = \frac{2}{3},\ x_2 = -2$$

$$f''(x) = \frac{\pm(-6x-4)2|x+1|^{\frac{1}{2}}(x^2+4)^2 \mp (-3x^2-4x+4)\cdot[2\frac{1}{2}|x+1|^{-\frac{1}{2}}(\pm 1)(x^2+4)^2 + 2|x+1|^{\frac{1}{2}}2(x^2+4)\cdot 2x]}{4|x+1|(x^2+4)^4}$$

$$= \frac{\mp \cancel{4}(3x+2)\sqrt{|x+1|}\cancel{(x^2+4)^2}}{\cancel{4}|x+1|(x^2+4)^{\cancel{4}\,2}} \pm \frac{(3x^2+4x-4)\left[\pm\frac{(x^2+4)^2}{\sqrt{|x+1|}} + 8x\sqrt{|x+1|}(x^2+4)\right]}{4|x+1|(x^2+4)^4}$$

$$= \mp \frac{3x+2}{\sqrt{|x+1|}(x^2+4)^2} \pm \frac{(3x^2+4x-4)\sqrt{|x+1|}\cancel{(x^2+4)}\left(\pm\frac{x^2+4}{|x+1|} + 8x\right)}{4|x+1|(x^2+4)^{\cancel{4}\,3}}$$

$$= \frac{3x^2+4x-4}{4\sqrt{|x+1|}(x^2+4)^3}\cdot\left(\frac{x^2+4}{|x+1|} \pm 8x\right) \mp \frac{3x+2}{\sqrt{|x+1|}(x^2+4)^2}$$

$f''(\frac{2}{3}) < 0 \Rightarrow$ lokales Maximum an $x = \frac{2}{3}$.

$f''(-2) < 0 \Rightarrow$ lokales Maximum an $x = -2$.

<u>Betrachte $x = -1$:</u>

$$\lim_{x\to -1^+} f'(x) = \frac{-3+4+4}{0} = \frac{5}{0} = +\infty$$

$$\lim_{x\to -1^-} f'(x) = -\frac{5}{0} = -\infty$$

$\Rightarrow$ lokales Minimum an $x = -1$.

392)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x^3$$

(*)

1. TP: $\sum_{n=0}^{0} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x = x^3$

$\Leftrightarrow x^2 = 1$

$\Leftrightarrow x = \pm 1 \Rightarrow x_1 = 1$ (lt. Angabe)

Sgn Fehler (da alt. Reihe) ⊖

2. TP: $\sum_{n=0}^{1} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 = x^3$

$\Leftrightarrow x = \frac{7}{6}x^3$

$\Leftrightarrow x = \pm\sqrt{\frac{6}{7}} \Rightarrow x_2 = \sqrt{\frac{6}{7}}$ ⊕

3. TP: $\sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 = x^3$

$\Leftrightarrow 1 - \frac{7}{6}x^2 + \frac{1}{120}x^4 = 0$

$y := x^2 \quad \Leftrightarrow \frac{1}{120}y^2 - \frac{7}{6}y + 1 = 0$

$\Leftrightarrow y^2 - \frac{\cancel{120}^{20} \cdot 7}{\cancel{6}} y + 120 = 0$

$\Leftrightarrow y_{1,2} = 70 \pm \sqrt{70^2 - 120}$

$= 70 \pm \sqrt{4780} \overset{(*)}{\Rightarrow} x_3 = \sqrt{70 - \sqrt{4780}}$ ⊖

(*) Beh.: Keine Lösung erfüllt $x > 1$.

Bew.: $|\sin x| \leq 1 \ \forall x \in \mathbb{R} \Rightarrow |x^3| \leq 1 \Leftrightarrow |x| \leq 1$