24.3.2010

Angewandte Statistik UE

I, 1, Box-Muller-Methode

a) U_1, U_2 i.i.d ; $U_i \sim U_{0,1} \Rightarrow f(x) = 1 \cdot \mathbb{1}_{(0,1)}(x)$

<u>ZZ:</u> $X = \sqrt{-2 \ln U_1} \cos 2\pi U_2$

$Y = \sqrt{-2 \ln U_1} \sin 2\pi U_2 \quad \sim N(0,1)$ und unabhängig.

Definiere $X = R \cos \Phi$ mit $R = \sqrt{-2 \ln U_1}$

$Y = R \sin \Phi \qquad \Phi := 2\pi U_2$

$$\mathrm{Det}\left(\frac{d(x,y)}{d(u_1,u_2)}\right) = \mathrm{Det}\left(\frac{d(x,y)}{d(r,\varphi)} \cdot \frac{d(r,\varphi)}{d(u_1,u_2)}\right) = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} \cdot \begin{vmatrix} -(\ln U_1)^{-\frac{1}{2}} \cdot \frac{1}{U_1} & 0 \\ 0 & 2\pi \end{vmatrix}$$

$$= \cancel{r}\, 2\pi \frac{1}{\cancel{\sqrt{-2\ln u_1}}} \frac{1}{u_1} = -\frac{2\pi}{u_1} \neq 0$$

$$\begin{cases} X^2 + Y^2 = R^2 = -2\ln U_1 & \Rightarrow U_1 = \exp\left[-\frac{1}{2}(X^2+Y^2)\right] \\ \tan \Phi = \frac{Y}{X} & \Rightarrow U_2 = \frac{1}{2\pi} \arctan \frac{Y}{X} \end{cases}$$

$$\mathrm{Det}\left(\frac{d(u_1,u_2)}{d(x,y)}\right) = \begin{vmatrix} \exp[\ldots]\cdot(-x) & \exp[\ldots]\cdot(-y) \\ -\frac{1}{2\pi}\frac{1}{1+(\frac{x}{y})^2}\frac{y}{x^2} & \frac{1}{2\pi}\frac{1}{1+(\frac{x}{y})^2}\frac{1}{x} \end{vmatrix}$$

$$= \frac{1}{2\pi}\exp[\ldots]\frac{1}{1+(\frac{x}{y})^2}\left(-1-\frac{y^2}{x^2}\right)$$

$$= -\frac{1}{2\pi}\exp\left[-\frac{1}{2}(x^2+y^2)\right]$$

<u>Transformationssatz:</u>

$$g(x,y) = f\Big(\underbrace{\exp[-\tfrac{1}{2}\underbrace{(x^2+y^2)}_{>0}]}_{\in(0,1)},\ \underbrace{\tfrac{1}{2\pi}\arctan \underbrace{\tfrac{y}{x}}_{>0}}_{\in(0,\frac{1}{4}]}\Big) \cdot \left|\mathrm{Det}\left(\frac{d(u_1,u_2)}{d(x,y)}\right)\right|$$

$$= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(x^2+y^2)\right] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$= g(x) \cdot g(y)$$

$\Rightarrow X, Y \sim N(0,1)$, unabh.

b) ZZ: $X' = (-2\ln U_1)^{\frac{1}{2}}\{\sqrt{1-\rho^2}\cos 2\pi U_2 + \rho \sin 2\pi U_2\}$
$Y' = (-2\ln U_1)^{\frac{1}{2}} \sin 2\pi U_2$ $\sim N_2(0,0,1,1,\rho)$

Definiere $X' = \sqrt{1-\rho^2}\, X + \rho \cdot Y$ mit X, Y aus a)
$Y' = Y$

$$\Rightarrow \begin{array}{l} Y = Y' \\ X = \frac{X' - \rho Y}{\sqrt{1-\rho^2}} \end{array}$$

$$\mathrm{Det}\left(\frac{\partial(x,y)}{\partial(x',y')}\right) = \begin{vmatrix} \frac{1}{\sqrt{1-\rho^2}} & -\frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{1-\rho^2}}$$

Transformationssatz:

$$h_{(X',Y')} = \frac{1}{\sqrt{1-\rho^2}}\, g_X\left(\frac{x'-\rho y}{\sqrt{1-\rho^2}}\right) \cdot g_Y(y')$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big[\underbrace{-\frac{1}{2}\frac{(x'-\rho y)^2}{1-\rho^2} - \frac{y'}{2}}_{-\frac{1}{2(1-\rho^2)}\left(x'^2 - 2x'y'\rho + \cancel{\rho^2 y'} + y' - \cancel{\rho^2 y'}\right)}\Big]$$

$\Rightarrow (X', Y') \sim N_2(0,0,1,1,\rho)$.

2 a) $X \sim Be_1(\alpha, \beta)$. ges.: $\mathbb{E}X$, $\mathrm{var}\, X$

$$\mathbb{E}X^n = \int_0^1 \frac{1}{B(\alpha,\beta)} \underbrace{x^{\alpha+n-1}}_{g} \underbrace{(1-x)^{\beta-1}}_{f} dx$$

$$= \frac{1}{B(\alpha,\beta)}\left(\underbrace{-\frac{1}{\beta} x^{\alpha+n-1}(1-x)^{\beta}\Big|_0^1}_{=0 \text{ für } n\neq 0} + \frac{\alpha+n-1}{\beta}\int_0^1 x^{\alpha+n-2} \overbrace{(1-x)^{\beta}}^{(1-x)^{\beta-1}(1-x)} dx\right)$$

$$= \frac{1}{B(\alpha,\beta)} \frac{\alpha+n-1}{\beta}\left(\int_0^1 x^{\alpha+n-2}(1-x)^{\beta-1}dx - \int_0^1 x^{\alpha+n-1}(1-x)^{\beta-1} dx\right)$$

$$= \frac{\alpha+n-1}{\beta}\left(\mathbb{E}X^{n-1} - \mathbb{E}X^n\right)$$

$$\Leftrightarrow \mathbb{E}X^n = \frac{\alpha+n-1}{\alpha+\beta+n-1}\mathbb{E}X^{n-1}$$

$$\Rightarrow \mathbb{E}X^1 = \frac{\alpha}{\alpha+\beta}\mathbb{E}1 = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\alpha+1}{\alpha+\beta+1}\frac{\alpha}{\alpha+\beta} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

b) $X \sim Gam(\alpha, \beta)$. ges.: $\mathbb{E}X$, var X

$$\mathbb{E}X^n = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \underbrace{x^{\alpha+n-1}}_{g} \underbrace{e^{-\frac{x}{\beta}}}_{f'} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\underbrace{-\beta x^{\alpha+n-1} e^{-\frac{x}{\beta}} \Big|_0^\infty}_{=0 \text{ für } n\neq 0} + \beta(\alpha+n-1)\int_0^\infty x^{\alpha+n} e^{-\frac{x}{\beta}} dx \right)$$

$$= \beta(\alpha+n-1)\cdot \mathbb{E}X^{n-1}$$

$$\Rightarrow \mathbb{E}X^1 = \alpha\beta$$

$$\text{var } X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}X\,(\beta(\alpha+1) - \mathbb{E}X) = \alpha\beta^2.$$

3) $X \sim Gam(\alpha_1, \beta)$, $Y \sim Gam(\alpha_2, \beta)$ ua.

<u>ZZ:</u> $X+Y \sim Gam(\alpha_1+\alpha_2, \beta)$

Faltung:

$$f_{X+Y}(s) = \int_0^s f_X(t)\cdot f_Y(s-t)\,dt$$

$$= \int_0^s \frac{1}{\Gamma(\alpha_1)\beta^{\alpha_1}} t^{\alpha_1-1} e^{-\frac{t}{\beta}} \cdot \frac{1}{\Gamma(\alpha_2)\beta^{\alpha_2}} (s-t)^{\alpha_2-1} e^{-\frac{s-t}{\beta}}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} e^{-\frac{s}{\beta}} \int_0^s t^{\alpha_1-1}(s-t)^{\alpha_2-1}\,dt$$

$$\int_0^s t^{\alpha_1-1}(s-t)^{\alpha_2-1}\,dt = \left| \begin{matrix} u = \frac{t}{s} \\ du = \frac{1}{s}dt \end{matrix} \right| \overset{t=su}{=} \int_0^1 s\,(su)^{\alpha_1-1}(s-su)^{\alpha_2-1}\,du$$

$$= s^{\alpha_1+\alpha_2-1}\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du$$

$$= s^{\alpha_1+\alpha_2-1}\cdot B(\alpha_1,\alpha_2)$$

$$\Rightarrow f_{X+Y}(s) = \frac{\overset{\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}}{\overset{\shortparallel}{B(\alpha_1,\alpha_2)}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} s^{\alpha_1+\alpha_2-1} e^{-\frac{s}{\beta}} = \frac{1}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} s^{\alpha_1+\alpha_2-1} e^{-\frac{s}{\beta}}$$

$$\Rightarrow X+Y \sim Gam(\alpha_1+\alpha_2, \beta)$$

4) $G(k) = \int_p^1 \frac{n!}{k!\,(n-(k+1))!} \, z^k (1-z)^{n-(k+1)} \, dz \qquad (k \in \{0, \dots, n-1\})$

(i) <u>ZZ:</u> $G(k) = \sum_{x=0}^{k} \binom{n}{x} p^x (1-p)^{n-x}$

$$G(k) = \int_p^1 \frac{n!}{k!\,(n-k-1)!} \underbrace{z^k}_{g} \underbrace{(1-z)^{n-k-1}}_{f'} \, dz$$

$$= \frac{n!}{k!\,(n-k-1)!} \left(- \frac{1}{n-k} z^k (1-z)^{n-k} \Big|_p^1 + \frac{k}{n-k} \int_p^1 z^{k-1} (1-z)^{n-k} \, dz \right)$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} + \frac{n!}{(k-1)!\,(n-k)!} \int_p^1 z^{k-1} (1-z)^{n-k} \, dz$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + G(k-1)$$

$$\Rightarrow G(k) = \sum_{x=0}^{k} \binom{n}{x} p^x (1-p)^{n-x}$$

(ii) $$\frac{n!}{(k-1)!\,(n-k)!} = \frac{\Gamma(\overbrace{n+1}^{n-k+1-k})}{\Gamma(k)\,\Gamma(n-k+1)} = \frac{1}{\cancel{B(\cdot,\cdot)}\, B(k, n-k+1)}$$

$$\Rightarrow P([Z > p]) = P([X \le k-1]) \quad \text{für } Z \sim Be_1(k, n-k+1)$$
$$X \sim B_{n,p}$$

5) $X \sim N(\mu, \sigma^2)$

ZZ: $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi_1^2$

$Y := \left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$, also ZZ: $Y^2 =: Z \sim \chi_1^2$.

<u>Transformationssatz:</u>

$$f_Z(z) = f_Y(\sqrt{z}) \left| \frac{1}{2\sqrt{z}} \right| + f_Y(-\sqrt{z}) \cdot \left| - \frac{1}{2\sqrt{z}} \right| \qquad (z > 0)$$

$$= 2 f_Y(\sqrt{z}) \frac{1}{2\sqrt{z}} = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z}{2}\right] \cdot \frac{1}{\sqrt{z}}$$

$$= \frac{1}{\sqrt{2}\,\Gamma(\frac{1}{2})} \, z^{-\frac{1}{2}} e^{-\frac{z}{2}}$$

$\Rightarrow Z \sim \chi_1^2$.

6) $G(k) = \int_\mu^\infty \frac{1}{\Gamma(k+1)} z^k e^{-z}\,dz$ $\qquad (k \in \mathbb{N}_0)$

(i) <u>ZZ:</u> $G(k) = \sum_{x=0}^{k} \frac{\mu^x}{x!} e^{-\mu}$

$$G(k) = \int_\mu^\infty \frac{1}{k!} \underbrace{z^k}_{g} \underbrace{e^{-z}}_{f'} \,dz$$

$$= \frac{1}{k!}\left(-z^k e^{-z} \Big|_\mu^\infty + k \int_\mu^\infty z^{k-1} e^{-z}\,dz \right)$$

$$= \frac{1}{k!} \mu^{k} e^{-\mu} + \frac{1}{(k-1)!} \int_\mu^\infty z^{k-1} e^{-z}\,dz$$

$$= \frac{\mu^k}{k!} e^{-\mu} + G(k-1)$$

$$\Rightarrow G(k) = \sum_{x=0}^{k} \frac{\mu^x}{x!} e^{-\mu}$$

(ii) $P([Z > \mu]) = P([X \le k-1])$ für $Z \sim Gam(k,1)$, $X \sim P_\mu$

7) a) $X \sim Be_1(m,n)$

<u>ZZ:</u> $\frac{X/m}{(1-X)/n} \sim F_{2m,2n}$

$X \sim Be_1(m,n) \Rightarrow Y := \frac{X}{1-X} \sim Be_2(m,n)$; ZZ: $Y \frac{n}{m} =: Z \sim F_{2m,2n}$

$y = z \cdot \frac{m}{n} \Rightarrow \frac{dy}{dz} = \frac{m}{n}$

<u>Transformationssatz:</u>

$$f_Z(z) = f_Y\left(\frac{m}{n} z\right) \cdot \frac{m}{n} = \frac{1}{B(m,n)} \frac{\left(\frac{m}{n} z\right)^{m-1}}{\left(1 - \frac{m}{n} z\right)^{m+n}} \frac{m}{n}$$

$$= \frac{1}{B(m,n)} \frac{m^{m-1} n^{1-m} z^{m-1}}{(n - mz)^{m+n}} \frac{n^{n+m} \cdot m}{n}$$

$$= \frac{1}{B(m,n)} \frac{(2m)^m (2n)^n z^{m-1}}{(2n - 2mz)^{m+n}}$$

$$\Rightarrow Z \sim F_{2m,2n}$$

b) $\alpha = P([Z \le z_\alpha]) = P([Y \le \frac{m}{n} z_\alpha]) \overset{X = \frac{Y}{1+Y}}{=} P\left(\left[X \le \frac{\frac{m}{n} z_\alpha}{1 + \frac{m}{n} z_\alpha}\right]\right) = P\left(\left[X \le \underbrace{\frac{m z_\alpha}{n + m z_\alpha}}_{=: x_\alpha}\right]\right)$

c) $X \sim Be_1(2,5) \Rightarrow x_{0,95} = \frac{2 z_{0,95}}{5 + 2 z_{0,95}} \approx 0{,}582$ mit $Z \sim F(4,10)$

$X \sim Be_1(5,20) \Rightarrow x_{0,05} \approx 0{,}086$; $X \sim Be_1(50,15) \Rightarrow x_{0,9} \approx 0{,}834$