28.4.10

Angewandte Statistik UE

IV)

1. $X_i \sim N(\mu, \sigma^2)$ i.i.d

$$S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}$$

<u>ges.:</u> $\mathbb{E} S_n$, $\mathrm{Var}\, S_n$

Wissen: $Z := \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}$, d.h. $f(x) = \frac{1}{\Gamma(\frac{n-1}{2})\, 2^{\frac{n-1}{2}}}\, x^{\frac{n-3}{2}}\, e^{-\frac{x}{2}}\, \mathbb{1}_{(0,\infty)}(x)$

Sei $S_n^{2} = g(Z) = \sqrt{\frac{Z\sigma^2}{n-1}}$

$$\Rightarrow \mathbb{E} S_n = \mathbb{E} g(Z) = \int_0^\infty g(z) f(z)\, dz = \int_0^\infty \sqrt{\frac{z\sigma^2}{n-1}}\, \frac{1}{\Gamma(\frac{n+1}{2})\, 2^{\frac{n-1}{2}}}\, z^{\frac{n-3}{2}} e^{-\frac{z}{2}}\, dz$$

$$= \frac{\sigma}{\sqrt{n-1}} \int_0^\infty \frac{1}{\Gamma(\frac{n-1}{2})\, 2^{\frac{n-1}{2}}}\, z^{\frac{n}{2}-1} e^{-\frac{z}{2}}\, dz$$

$$= \frac{\sigma}{\sqrt{n-1}}\, \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\, \frac{2^{\frac{n}{2}}}{2^{\frac{n-1}{2}}} \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{n}{2})\, 2^{\frac{n}{2}}}\, z^{\frac{n}{2}-1} e^{-\frac{z}{2}}\, dz}_{=1} = \frac{\sqrt{2}}{\sqrt{n-1}}\, \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\, \sigma$$

$$\mathrm{Var}\, S_n = \mathbb{E} S_n^2 - (\mathbb{E} S_n)^2 = \sigma^2 \left(1 - \frac{\sqrt{2}}{\sqrt{n-1}}\, \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right).$$

2. $X \sim B_{n,\theta}$, d.h. $P([X=k]) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$ $\quad k \in \{0,\dots,n\}$

<u>ZZ:</u> $\nexists$ unverzerrte Schätzfkt. für $\frac{1}{\theta}$.

Sei $g(x)$ unverzerrte Schätzfkt. für $\frac{1}{\theta}$ mit $x \in \{0,\dots,n\}$. $\Rightarrow \mathbb{E}[g(x)] = \frac{1}{\theta}$ $\forall \theta \in (0,1)$.

$$\mathbb{E} g(x) = \sum_{k=0}^{n} g(k)\, P([X=k]) = \sum_{k=0}^{n} g(k) \binom{n}{k} \theta^k (1-\theta)^{n-k} = \frac{1}{\theta}$$

$$\Leftrightarrow \underbrace{\sum_{k=0}^{n} g(k) \binom{n}{k} \theta^{k+1} (1-\theta)^{n-k}}_{\text{Polynom in } \theta} = \underbrace{1}_{\text{Konstante}} \qquad \forall \theta \in (0,1)$$

$\Rightarrow \nexists$ unverzerrter Schätzer.

3. $X_1, \dots, X_n$ Stichprobe; $X_i \sim Ex_\lambda \Rightarrow f(x|\lambda) = \lambda e^{-x\lambda} \mathbb{1}_{(0,\infty)}(x)$

a) ges.: plausibler Schätzer von λ.

$$\ell(\lambda; x_1, \dots, x_n) = \prod_{i=1}^n \lambda e^{-x_i \lambda} = \lambda^n e^{-\lambda \sum x_i}$$

$$\tilde{\ell}(\lambda, x) = \ln(\ell(\lambda, x)) = n \ln \lambda - \lambda \sum_{i=1}^n x_i$$

$$\frac{\partial \tilde{\ell}}{\partial \lambda}(\lambda, x) = \frac{n}{\lambda} - \sum_{i=1}^n x_i = 0 \quad \Leftrightarrow \quad \hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\overline{x}_n}$$

$\Rightarrow$ Likelihood-Schätzer: $\hat{\lambda} = \frac{1}{\overline{X}_n}$

b) Schätzer unverzerrt?

Wissen: $X_i \sim Ex_\lambda = Gam(1, \lambda) \Rightarrow \underbrace{\sum_{i=1}^n X_i}_{= X} \sim Gam(n, \lambda)$

$$\mathbb{E}\hat{\lambda} = \mathbb{E}\frac{1}{\overline{X}_n} = \mathbb{E}\frac{n}{\sum X_n} = n\,\mathbb{E}\frac{1}{X}$$

$$\mathbb{E}\frac{1}{X} = \int_{\mathbb{R}^+} \frac{1}{x} \frac{1}{\Gamma(n)\left(\frac{1}{\lambda}\right)^n} x^{n-1} e^{-\lambda x}\, dx = \frac{\lambda^n}{\Gamma(n)} \int_{\mathbb{R}} x^{n-2} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^n}{\underbrace{\Gamma(n)}_{(n-1)\Gamma(n-1)}} \frac{\Gamma(n-1)}{\lambda^{n-1}} \underbrace{\int_{\mathbb{R}^+} \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda x} x^{(n-1)-1}\, dx}_{=1} = \frac{\lambda}{n-1}$$

$$\Rightarrow \mathbb{E}\hat{\lambda} = n\,\mathbb{E}\frac{1}{X} = \frac{n}{n-1}\lambda \neq \lambda, \text{ also verzerrt.}$$

Erwartungstreuer Schätzer für λ: $\hat{\lambda} = \frac{n-1}{\sum_{i=1}^n X_i}$.

c) ges.: Varianz des Schätzers

$$\mathbb{E}\frac{n-1}{\sum X_i} = (n-1)\,\mathbb{E}\frac{1}{\sum X_i} = \lambda.$$

$$\mathbb{E}\left(\frac{n-1}{\sum X_i}\right)^2 = (n-1)^2\,\mathbb{E}\frac{1}{X^2} = \frac{n-1}{n-2}\lambda^2$$

$$\mathbb{E}\frac{1}{X^2} = \int_{\mathbb{R}^+} \frac{1}{x^2} \dots = \frac{\lambda^2}{(n-1)(n-2)}$$

$$\Rightarrow \operatorname{Var}\hat{\lambda} = \mathbb{E}\hat{\lambda}^2 - (\mathbb{E}\hat{\lambda})^2 = \lambda^2\left(\frac{n-1}{n-2} - 1\right) = \frac{\lambda^2}{n-2}.$$

4. $X_i \sim \text{Gam}(\alpha,\beta)$, d.h. $f(x) = \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\,\beta^\alpha} \mathbb{1}_{(0,\infty)}(x)$

a) Bestimmung der „Momentenschätzer".

empr. 1. Stichprobenmoment: $\frac{1}{n}\sum x_i = \frac{185}{20} = \frac{37}{4}$

–"– 2. ——"——: $\frac{1}{n}\sum x_i^2 = \frac{1875}{20} = \frac{375}{4}$

$$\mathbb{E}\left[\frac{1}{n}\sum X_i\right] = \mathbb{E}X = \alpha\cdot\beta$$

$$\mathbb{E}\left[\frac{1}{n}\sum X_i^2\right] = \mathbb{E}X^2 \Rightarrow \operatorname{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \alpha\beta^2$$

$$\Rightarrow \begin{cases} \hat\alpha\cdot\hat\beta = \frac{37}{4} \\ \hat\alpha\cdot\hat\beta^2 = \frac{131}{16} \end{cases} \Rightarrow \begin{array}{l} \hat\alpha = \frac{1369}{131} \approx 10{,}45\ldots \\ \hat\beta = \frac{131}{148} \approx 0{,}885\ldots \end{array}$$

Ob Momente um 0 oder zentrale Momente betrachtet werden, spielt keine Rolle.

b) Bestimmung der Plausibilitätsbedingungen.

$$\ell(\alpha,\beta;x_1,\ldots,x_n) = \prod_{i=1}^n \frac{x_i^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x_i}{\beta}} = (\Gamma(\alpha)\beta^\alpha)^{-n}\left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\sum_{i=1}^n x_i \frac{1}{\beta}}$$

$$\Rightarrow \tilde\ell(\alpha,\beta,x) = -n\ln\Gamma(\alpha) - \alpha n\ln\beta + (\alpha-1)\sum_{i=1}^n \ln x_i - \sum_{i=1}^n x_i \frac{1}{\beta}$$

•) $\dfrac{\partial\tilde\ell}{\partial\beta}(\alpha,\beta,x) = -\dfrac{\alpha}{\beta}n + \sum_{i=1}^n x_i \dfrac{1}{\beta^2} = 0 \Leftrightarrow \beta = \dfrac{\sum x_i}{\alpha n} = \alpha\cdot\overline{x}_n$

•) $\dfrac{\partial\tilde\ell}{\partial\alpha}(\alpha,\beta,x) = -\dfrac{n}{\Gamma(\alpha)}\dfrac{\partial}{\partial\alpha}\Gamma(\alpha) - n\ln\beta + \sum_{i=1}^n \ln x_i = 0$

$$\Leftrightarrow \beta = e^{-\frac{\Gamma(\alpha)}{\Gamma'(\alpha)}} \sqrt[n]{\prod x_i}$$

$\Rightarrow \nexists$ explizite Lösung.

5. $X_1,\ldots,X_n$ Stichprobe $X \sim N(\mu,\sigma^2)$

<u>ges.:</u> plausibler Schätzer von x_γ

$$\widehat{(\mu,\sigma^2)} = \left(\overline{X}_n, \frac{n-1}{n}S_n^2\right) \qquad X\sim N(\mu,\sigma^2) \Rightarrow Z := \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$\Downarrow$$

$$X = \sigma Z + \mu$$

$$\gamma = P([X \le x_\gamma]) = P([\sigma Z+\mu \le x_\gamma]) = P([Z \le \underbrace{\tfrac{x_\gamma-\mu}{\sigma}}_{z_\gamma}]) = \Phi\left(\frac{x_\gamma-\mu}{\sigma}\right) \Leftrightarrow \frac{x_\gamma-\mu}{\sigma} = z_\gamma$$

$$\Rightarrow \hat{x}_\gamma = \hat\mu + \hat\sigma\cdot z_\gamma$$

golf.txt: $\hat\mu = 274{,}476 = \overline{x}_n$ $\quad z_{0,95} = 1{,}6448\ldots \Rightarrow \hat{x}_{0,95} = 288{,}1$

$\hat\sigma = \sqrt{\frac{99}{100}S_n^2} = 8{,}283\ldots$

6. $X_1,\dots,X_n$ Stichprobe; $X \sim U_{(a-\frac{b}{2},\, a+\frac{b}{2})}$ d.h. $f(x) = \frac{1}{b}\mathbb{1}_{(a-\frac{b}{2},\, a+\frac{b}{2})}(x)$

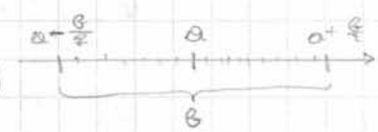
$$\ell(a,b;\, x_1,\dots,x_n) = \prod_{i=1}^{n}\left(\frac{1}{b}\right)^{\cancel{1}} \mathbb{1}_{(a-\frac{b}{2},\, a+\frac{b}{2})} = \begin{cases}\left(\frac{1}{b}\right)^n & x_i \in (a-\frac{b}{2},\, a+\frac{b}{2})\ \forall i \\ 0 & \text{sonst}\end{cases}$$

ℓ unstetig, also liefert herkömmliche Methode keine Lösung.

Ziel: $\frac{1}{b^n}$ max. $\Leftrightarrow$ b min. s.t. $b \geq \max x_i - \min x_i$

$$\Rightarrow \hat{b} = \max x_i - \min x_i = X_{(n)} - X_{(1)}$$

$$\Rightarrow \hat{a} = \frac{\max x_i + \min x_i}{2} = \frac{1}{2}\left(X_{(n)} + X_{(1)}\right)$$



Schätzer unverzerrt?

$\mathbb{E}\hat{a} = \mathbb{E}\left[\frac{X_{(1)}+X_{(n)}}{2}\right] = a$ (Symmetrie)

$\mathbb{E}\hat{b} = \mathbb{E}X_{(n)} - \mathbb{E}X_{(1)}$

$$F_{X_{(n)}}(x) = F(x)^n \Rightarrow f_{X_{(n)}}(x) = n\cdot F(x)^{n-1}\cdot f(x) = n\left(\frac{x-a+\frac{b}{2}}{b}\right)^{n-1}\frac{1}{b}\,\mathbb{1}_{(a-\frac{b}{2},\, a+\frac{b}{2})}(x)$$

$$\Rightarrow \mathbb{E}X_{(n)} = \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} x\, f_{X_{(n)}}(x)\, dx = \left(a+\frac{b}{2}\right) - \frac{b}{n+1}$$

$$\Rightarrow \mathbb{E}X_{(1)} = \overset{a-(X_{(n)}-a)}{2a - X_{(n)}} = \left(a-\frac{b}{2}\right) + \frac{b}{n+1}$$

$$\Rightarrow \mathbb{E}\hat{b} = \cancel{a} + \frac{b}{2} - \frac{b}{n+1} - \cancel{a} + \frac{b}{2} - \frac{b}{n+1} = b - \frac{2b}{n+1} = b\,\frac{n-1}{n+1} \neq b$$

$\tilde{b} := \frac{n+1}{n-1}\left(X_{(n)} - X_{(1)}\right)$ ist erwartungstreu.

Bsp.: $x_{(1)} = 2{,}91$, $x_{(n)} = 5{,}12$ $\Rightarrow$ $\tilde{b} = \cancel{2{,}24}\ \frac{6}{4}\ 2{,}21 = 3{,}315$

$\hat{a} = 4{,}015$