25.01.10

Maß u. Wahrscheinlichkeitsth. UE

XIII)

1) $(\Omega, \mathcal{F}, P)$

a) ZZ: $\left(\mathbb{E}|X|^\alpha\right)^{\frac{1}{\alpha}} \le \left(\mathbb{E}|X|^\beta\right)^{\frac{1}{\beta}}$ (Ljapunoff-Ungl.)

Hölder-Ungl.: $\frac{1}{r}+\frac{1}{q}=1$, $f^r \in \mathcal{L}$, $g^r \in \mathcal{L}$

$\Rightarrow \int |fg|\,d\mu \le \left(\int |f|^r d\mu\right)^{\frac{1}{r}} \left(\int |g|^q d\mu\right)^{\frac{1}{q}}$

$f := |X|^\alpha$, $g := 1$, $\mu := P$

$\rightsquigarrow \int \underbrace{||X|^\alpha|}_{=|X|^\alpha}\,dP \le \left(\int ||X|^\alpha|^r dP\right)^{\frac{1}{r}} \left(\int |1|^q dP\right)^{\frac{1}{q}}$

$\Leftrightarrow \quad \mathbb{E}|X|^\alpha \le \left(\mathbb{E}|X|^{\alpha r}\right)^{\frac{1}{r}} \underbrace{\left(\mathbb{E}1\right)^{\frac{1}{q}}}_{=1}$

$r := \frac{\beta}{\alpha} \rightsquigarrow \mathbb{E}|X|^\alpha \le \left(\mathbb{E}|X|^\beta\right)^{\frac{\alpha}{\beta}}$

$\Leftrightarrow \left(\mathbb{E}|X|^\alpha\right)^{\frac{1}{\alpha}} \le \left(\mathbb{E}|X|^\beta\right)^{\frac{1}{\beta}}$

b) ZZ: $(\mathbb{E}X)^{-r} \le \mathbb{E}(X^{-r})$ $\qquad X>0,\ r>0$

$\varphi(x) = x^{-r} \Rightarrow \varphi''(x) = r(r+1)\,x^{-r-2} > 0 \quad \forall x>0$

Jensen-Ungl. $\Rightarrow \varphi(\mathbb{E}X) \le \mathbb{E}(\varphi(X))$

$\Leftrightarrow (\mathbb{E}X)^{-r} \le \mathbb{E}(X^{-r})$.

2) $X \sim \text{Cauchy}(a)$, d.h. $f(x) = \frac{a}{\pi(a^2+x^2)}$ $\quad (a>0)$

$\mathbb{E}X^+ = \int_{\mathbb{R}^+} \frac{ax}{\pi(a^2+x^2)}\,d\lambda(x) = \frac{a}{\pi}\int_{\mathbb{R}^+} \frac{x}{a^2+x^2}\,d\lambda(x) = \frac{1}{\pi}\int_{\mathbb{R}^+} \frac{\left(\frac{x}{a}\right)}{1+\left(\frac{x}{a}\right)^2}\,d\lambda(x)$

$= \frac{1}{\pi}\int_{\mathbb{R}^+} \frac{y}{1+y^2}\,d\lambda(y) \ge \frac{1}{\pi}\int_{[1,\infty)} \frac{y}{1+y^2}\,d\lambda(y) \ge \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{n}{1+n^2} \ge \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{n}{2n^2} = \infty$.

$\mathbb{E}X^- = \infty$ (analog) $\Rightarrow \mathbb{E}X \nexists$

3) $X_1, \dots, X_n$ i.i.d. $\quad X \sim A_{1/2}$

$R(\omega)$ ... Anz. der Runs

$(X_{i+1}(\omega), \dots, X_{i+k}(\omega))$ Run der Länge k, wenn gilt:

- $i = 0 \vee X_i(\omega) \neq X_{i+1}(\omega)$
- $X_{i+1}(\omega) = \dots = X_{i+k}(\omega)$
- $i + k = n \vee X_{i+k}(\omega) \neq X_{i+k+1}(\omega)$

a) Bilde Folge Y_i mit $Y_i(\omega) = \begin{cases} 1 & X_i(\omega) \neq X_{i-1}(\omega) \\ 0 & \text{sonst} \end{cases} \quad i = 2, 3, \dots, n$

$$Y_1(\omega) = 1$$

$$\Rightarrow R(\omega) = \sum_{i=1}^{n} Y_i(\omega)$$

$$\Rightarrow \mathbb{E}R = \sum_{i=1}^{n} \mathbb{E}Y_i = 1 + \sum_{i=2}^{n} P([X_i \neq X_{i-1}])$$
$$= 1 + \sum_{i=2}^{n} P([(X_i = 1 \wedge X_{i-1} = 0) \vee (X_i = 0 \wedge X_{i-1} = -1)])$$
$$= 1 + \sum_{i=2}^{n} 2 \frac{1}{2^2} = 1 + \frac{1}{2}(n-1) = \frac{n+1}{2}.$$

$$\operatorname{cov}(Y_i, Y_j) = \mathbb{E}(Y_i \cdot Y_j) - (\mathbb{E}Y_i \cdot \mathbb{E}Y_j) = 0 \qquad i \neq j \wedge i, j \in \{1, \dots, n\}$$
$$\mathbb{E}(Y_i Y_j) = P([Y_i = 1, Y_j = 1]) = \frac{1}{4} = \mathbb{E}Y_i \cdot \mathbb{E}Y_j$$

$$\Rightarrow \operatorname{var} R = \sum_{i=1}^{n} \operatorname{var}(Y_i) = \underbrace{\operatorname{var} Y_1}_{0} + \sum_{i=2}^{n} \operatorname{var} Y_i = \frac{n-1}{4}.$$

b) $n = 50 \Rightarrow \mathbb{E}R = 25{,}5$; $\operatorname{var} R =: \sigma^2 = \frac{49}{4} \Rightarrow \sigma = \frac{7}{2} = 3{,}5$

$R(\omega) = 36$

$|R - \mathbb{E}R| \geq \lambda\sigma \Leftrightarrow |36 - 25{,}5| = 10{,}5 \geq \lambda \cdot 3{,}5 \Leftrightarrow \lambda \leq 3$

Tschebyscheffsche Ungl.:

$$P(|R - \mathbb{E}R| \geq \lambda\sigma) \leq \frac{1}{\lambda^2} \Rightarrow P(|R - \mathbb{E}R| \geq 3\sigma) \leq \frac{1}{9}.$$

4) $f: [a,b] \to \mathbb{R}$

a) $f(x) = x \cdot \sin\frac{1}{x} \qquad [a,b] = [0, \frac{1}{\pi}]$

$|f(x)| = 0 \Leftrightarrow x = 0 \vee x = \frac{1}{k\pi}$

$|f(x)| = x \Leftrightarrow x = \frac{1}{k\pi + \frac{\pi}{2}}$

Bilde Teilungspunktfolge

$x_n = \frac{1}{(n+1)\frac{\pi}{2}} = \frac{2}{(n+1)\pi} \qquad n \in \{1, \dots, N-1\}$

$x_N = 0$

$$\sum_{n=1}^{N-1} |f(x_{n+1}) - f(x_n)| = \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor} 2 \cdot x_{2k} = \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor} \frac{4}{(2k+1)\pi}$$

$$= \frac{4}{\pi} \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor} \frac{1}{2k+1} \longrightarrow \infty \text{ für } N \to \infty$$

$\Rightarrow V_0^{\frac{1}{\pi}} f = \infty.$

b) $f(x) = F_c(x)$ (Cantor-Funktion) $\quad [a,b] = [0,1]$

$$1 = F_c(1) - F_c(0) = \int_{[0,1]} F'(t)\, d\lambda(t)$$

$$= \int_{[0,1]\setminus C} F'(t)\, d\lambda(t) = 0.$$

$\Rightarrow F_c$ stetig und monoton, aber nicht abs. stetig.

c) $$\sum_{i=1}^{n} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq C \cdot \underbrace{\sum_{i=1}^{n} (b_i - a_i)}_{<\delta} < \varepsilon$$ für $\varepsilon := C \cdot \delta$.

d) 1. MWS der Diff.-Rechnung:

$\exists x_0 \in [a,b]$ mit $f'(x_0) = \frac{f(b)-f(a)}{b-a} \Leftrightarrow f(b) - f(a) = f'(x_0)(b-a)$

$$\Rightarrow |f(b) - f(a)| \leq \underbrace{\sup_{x \in [a,b]} |f'(x)|}_{=:C} \cdot |b-a|$$

$\Rightarrow f$ abs. stetig lt. c)

5) (X_n) unabh.

$$P([X_n = n]) = P([X_n = -n]) = \frac{1}{2n \log n}$$

$$P([X_n = 0]) = 1 - \frac{1}{n \log n} \qquad n \geq 2$$

(i) ZZ: $\overline{X_n} := \frac{1}{n} \sum_{i=2}^{n} X_i \to 0$ in WS

$$\mathbb{E}X_n = n \frac{1}{2n \log n} - n \frac{1}{2n \log n} + 0 \cdot \left(1 - \frac{1}{n \log n}\right) = 0 \quad \forall n \geq 2$$

Schw. Gesetz d. gr. Zahlen $\Rightarrow \lim_{n \to \infty} P([\overline{X}_n > \varepsilon]) = 0 \quad \forall \varepsilon > 0.$

(ii) ZZ: $\overline{X_n} \not\to 0$ P-fs.

6) $X_n \sim Ex_{\lambda_n} \Rightarrow \mathbb{E}X_n = \frac{1}{\lambda_n} \quad \forall n \in \mathbb{N}. \qquad X := \sum_{n=1}^{\infty} X_n$

a)

$$\mathbb{E}|X| = \mathbb{E}X = \sum_{n=1}^{\infty} \mathbb{E}X_n = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$$

$$\overset{(15.37)}{\Rightarrow} \sum_{k=1}^{\infty} P([|X| > k]) < \infty \Rightarrow \lim_{k \to \infty} P([|X| > k]) = 0$$

$$\Rightarrow X < \infty \text{ P-fs.}$$

b)

$$\mathbb{E}|X| = \mathbb{E}X = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} P([X > k]) = \infty \overset{\text{2. Borel-Cantelli-Lemma}}{\Rightarrow} P(\limsup_k [X > k]) = 1$$

$$\Rightarrow P([X = \infty]) = 1, \text{ d.h. } X = \infty \text{ P-fs.}$$

7) a) Sei $X_i \in \{1, \dots, m\}$ das Ergebnis des i. Versuches.

$$S_1 = m \cdot b_{X_1} \cdot S_0$$

$$S_2 = m \cdot b_{X_2} \cdot S_1$$

$$\Rightarrow S_n = m^n \cdot S_0 \cdot \prod_{i=1}^{n} b_{X_i}$$

b) $$\frac{1}{n} \log S_n = \log m + \frac{\log S_0}{n} + \frac{1}{n} \sum_{i=1}^{n} \log b_{X_i}$$

GGZ: $$\lim_{n \to \infty} \left(\left[\left| \frac{1}{n} \sum_{i=1}^{n} \log b_{X_i} - \mathbb{E} \log b_{X_i} \right| > \varepsilon \right] \right) = 0$$

$$\Rightarrow \frac{1}{n} \log S_n \xrightarrow{P} \log m + \mathbb{E} \log b_{X_i}$$

c) $$\hat{S}_n = m^n \cdot S_0 \cdot \prod_{i=1}^{n} p_{X_i}$$

$$\Rightarrow \frac{1}{n} \log \hat{S}_n \xrightarrow{P} \log m + \mathbb{E} \log p_{X_i}$$

d) $$\sum_{i=1}^{m} p_i \log \overset{b_i}{\cancel{p_i}} - \sum_{i=1}^{m} p_i \log p_i = \sum_{i=1}^{m} p_i \log \frac{b_i}{p_i}$$

$$\leq \sum_{i=1}^{m} p_i \left(\frac{b_i}{p_i} - 1 \right) = \sum_{i=1}^{m} b_i - \sum_{i=1}^{m} p_i \leq 0.$$

e) aus d) folgt $\mathbb{E} \log p_{X_i} \geq \mathbb{E} \log b_{X_i}$, weshalb c) zu bevorzugen ist.

f) $$\mathbb{E} S_1 = m \cdot S_0 \cdot \mathbb{E} b_{X_1} = m \cdot S_0 \cdot \sum_{i=1}^{m} p_i b_i = m \cdot S_0 \cdot p_{i_0}.$$